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The Martinelli–Parisi systematic expansion in lattice gauge theory–$Z(2)$ model on a cubic lattice

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Abstract. The Martinelli-Parisi systematic expansion is applied to the three-dimensional $Z(2)$ system as a first application to gauge theory. To first order in perturbation, four kinds of new operators appear. The critical point is significantly improved, while the non-trivial fixed point acquires an undesirable relevant eigenvalue.

1. Introduction

The real space renormalisation group (RSRG) is a powerful method for studying critical phenomena. In recent works on lattice gauge theory, it has been shown that RSRG is very useful when it is combined with the Monte Carlo simulation techniques. On the other hand, it is also very important to exploit analytical tools to acquire a fundamental understanding of the theory. The Migdal-Kadanoff (MK) renormalisation group recursion equation [1, 2] has been found to provide one such tool [3-6].

Some years ago, Martinelli and Parisi showed how to improve systematically the MK equation by introducing a parameter ε, which formally connects the MK result ($\varepsilon = 0$) and the exact case ($\varepsilon = 1$) [7]. The ε is a quantity which controls a magnitude of the potential shifting. The advantage is that, by developing the physical quantities in powers of ε, one can systematically incorporate the long range interactions according to the powers of ε. Such a method has been successfully applied to two-dimensional systems such as the Ising model [7], the Potts model [8], the O(N) non-linear σ model [9] and the $Z(4)$ model [10]. So it would seem reasonable to apply this expansion method to lattice gauge theory in order to acquire accurate renormalisation group equations. Before studying more realistic systems like the 4D non-Abelian gauge theory, we consider here, as a first step, the simplest non-trivial lattice gauge system, namely the $Z(2)$ theory on a three-dimensional cubic lattice.

The following procedure is used to obtain the recursion equations. Starting from a three-dimensional cubic elementary cell of size L^3, we perform dilatation R_μ successively in the three directions ($\mu = x, y, z$) and get the effective theory on the cubic cell of size $(2L)^3$. Each R_μ consists of a potential shifting and a subsequent decimation. The shift in question is of the inner plaquettes of the obtained rectangular parallelepipeds to the outer ones by the amount $(1-\varepsilon)\beta_\mu\Delta_\mu$, where Δ_μ refers to the simple plaquettes perpendicular to the μ direction and β_μ are the respective coupling constants. The decimation is a summation over the link variables on the inner plaquettes with the strength $\varepsilon\beta_\mu$. After symmetrisation on the isotropy, such a process produces four new coupling constants of O(ε). These coupling constants are associated with the interactions of the plaquettes with the three-dimensional configurations.

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The result, which we will discuss in detail in § 3, is that the value of the critical coupling constant is remarkably improved—in fact it is almost equal to the known result—but the fixed point acquires an undesirable new relevant eigenvalue.

In the following section, we give the detailed derivation of the MK recursion equations for this case. In § 3, the results and some remarks are presented.

2. 3D $Z(2)$ model

We shall consider a three-dimensional $Z(2)$ lattice gauge theory with action

$$S = \beta \sum_{\langle ijkl \rangle} \sigma_i \sigma_j \sigma_k \sigma_l \tag{2.1}$$

where the σ are $Z(2)$ link variables on the contour $\langle ijkl \rangle$ of a plaquette and β is a coupling constant.

2.1. Simple example

In the first place, to make it clear how the recursion equations will be constructed, let us look at a simple calculation in which only a dilatation in the x direction is carried out and the theory has only a single coupling constant β.

Figure 1 shows a dilatation in the x direction. Inner plaquettes in the yz direction of the prolonged cell are shifted to the outer ones by the amount $\beta(1-\varepsilon)\Delta_x$, where Δ_x refers to the plaquettes in the yz direction.

The decimation is performed over the link variables of the inner plaquettes with strength $\varepsilon\beta$. We use the formula

$$\exp(\beta_p \sigma_p) = \cosh \beta_p (1 + \tanh \beta_p \sigma_p) \tag{2.2}$$

for each plaquette σ_p with the coupling constant β_p. By summing over the link

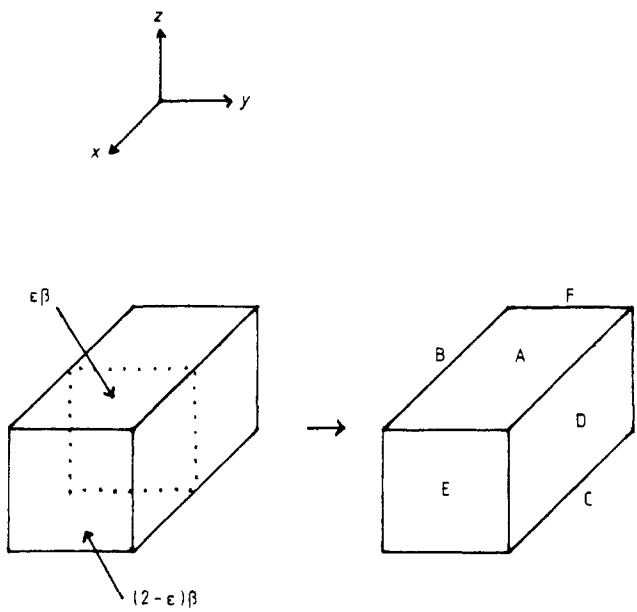

Figure 1. The dilatation R_x in the x direction.

variables on the internal plaquettes, we obtain, to $O(\varepsilon)$

$$\begin{aligned}\exp(S'_{\text{cell}}) = C\{&(1+p^2\sigma_A)(1+p^2\sigma_B)(1+p^2\sigma_C)(1+p^2\sigma_D)\\&+p^4 r[(\sigma_E+\sigma_F)+(\sigma_E+\sigma_F)(\sigma_A+\sigma_B+\sigma_C+\sigma_D)\\&+(\sigma_E+\sigma_F)(\sigma_A\sigma_B+\sigma_A\sigma_C+\sigma_A\sigma_D)]\}\exp[(2-\varepsilon)\beta(\sigma_E+\sigma_F)]\end{aligned} \tag{2.3}$$

where p and r are given by $p=\tanh\beta$ and $r=\tanh\varepsilon\beta\simeq\varepsilon\beta+O(\varepsilon^2)$, respectively, and C is some function of β, independent of the link variables. σ_A, σ_B, σ_C and σ_D denote the plaquette variables of the rectangles A, B, C and D, respectively, and σ_E and σ_F are plaquettes of the squares E and F (see figure 1). In order to get an effective action S'_{cell} of the new cells, the RHS of (2.3) is exponentiated by using the formula

$$(1+p^2\sigma)^{-1}=(1-p^2\sigma)/(1-p^4). \tag{2.4}$$

As a consequence, S'_{cell} is given by

$$\begin{aligned}S'_{\text{cell}} = &\beta'(\sigma_A+\sigma_B+\sigma_C+\sigma_D)+\tilde{\beta}'(\sigma_E+\sigma_F)\\&+\hat{\beta}'[(\sigma_E+\sigma_F)(\sigma_A+\sigma_B+\sigma_C+\sigma_D)\\&+(\sigma_E+\sigma_F)(\sigma_A\sigma_B+\sigma_A\sigma_C+\sigma_A\sigma_D)]+\text{constant}.\end{aligned} \tag{2.5}$$

We now find that in addition to the simple plaquette interactions $\sigma_A,\ldots,\sigma_F$, more complicated interactions of $O(\varepsilon)$ are generated, namely, $\sigma_E\sigma_A$, $\sigma_E\sigma_B$, ... (chair type), $\sigma_E\sigma_A\sigma_C$, ... (bridge type) and $\sigma_E\sigma_A\sigma_B$, ... (corner type). They are illustrated in figure 2. The new coupling constants β', $\tilde{\beta}'$ and $\hat{\beta}'$ are given by

$$\begin{aligned}&\tanh\beta'=(\tanh\beta)^2\\&\tanh\tilde{\beta}'=\tanh[(2-\varepsilon)\beta]+K\{1-\tanh^2[(2-\varepsilon)\beta]\}\\&\tanh\hat{\beta}'=K\\&K=\varepsilon\beta\frac{\tanh^4\beta}{(1+\tanh^2\beta)^4}.\end{aligned} \tag{2.6}$$

It is noted that the link variable appearing twice in the product of the plaquette

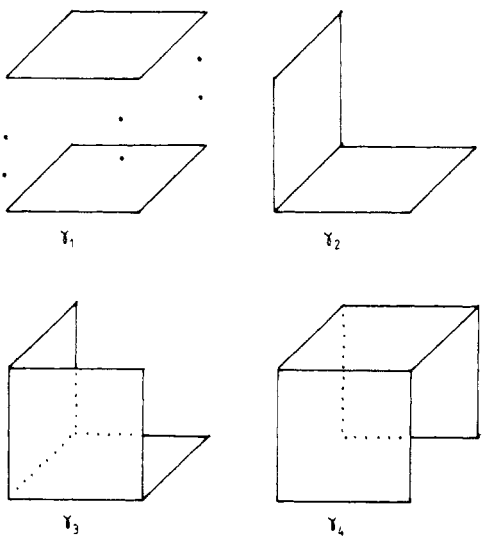

Figure 2. The plaquette interactions of $O(\varepsilon)$; parallel type (coupling constant $\varepsilon\gamma_1$), chair type ($\varepsilon\gamma_2$), corner type ($\varepsilon\gamma_3$) and bridge type ($\varepsilon\gamma_4$).

variables like $\sigma_E\sigma_A$ does not contribute due to the $Z(2)$ property of the variables. Moreover an identity on the product of six different plaquette variables $\sigma_A\sigma_B\sigma_C\sigma_D\sigma_E\sigma_F = 1$ is taken into account to obtain (2.6).

2.2. Recursion equations

If, starting from the single coupling constant β of the model (2.1) we had also made dilatations in the y and z directions, as well as the x direction, then we would have thirteen different coupling constants instead of the three coupling constants of (2.6). Thirteen is the number of all the possible different coupling constants generated to $O(\varepsilon)$. However, it would be reduced to five after isotropy symmetrisation.

Now, to get a complete recursion equation, we start from a theory with five coupling constants $(\beta, \varepsilon\gamma_1, \varepsilon\gamma_2, \varepsilon\gamma_3, \varepsilon\gamma_4)$ on a cube. The β is the coupling constant corresponding to the simple plaquettes, and the others of $O(\varepsilon)$ $\{\varepsilon\gamma_i\}$ are associated with the interactions of the plaquettes of the three-dimensional configurations as shown in figure 2.

As the algebraic calculation to get the recursion equations are beyond that by hand, we use the SCHOONSCHIP to obtain them. By taking the potential shifting of the type $\Delta_x\beta(1-\varepsilon)$, the dilatation R_x in the x direction provides the following transformations:

$$\begin{aligned}
&\tanh\beta'_A = \tanh^2\beta + K_x(1-\tanh^4\beta)\hat{\varepsilon}_1 \\
&\tanh\beta'_B = \tanh\beta'_A \\
&\tanh\beta'_E = \tanh[(2-\varepsilon)\beta] + K_x(1-\tanh^2[(2-\varepsilon)\beta])\hat{\varepsilon}_3 \\
&\tanh\delta'_\alpha = K_x\hat{\varepsilon}_{\alpha+3} \qquad (\alpha = 1\text{-}10) \\
&K_x = (1+\tanh^2\beta)^{-4}
\end{aligned} \tag{2.7}$$

where β'_A and β'_B denote the coupling constants of rectangular plaquettes A(C) and B(D), respectively, and β'_E is that of the square plaquettes E(F) in figure 1.

The new couplings of $O(\varepsilon)$ are referred to as δ'_α $(\alpha = 1\text{-}10)$ and the $O(\varepsilon)$ part of the new effective action is given by

$$\begin{aligned}
&\delta'_1\sigma_A\sigma_C + \delta'_2\sigma_B\sigma_D + \delta'_3\sigma_E\sigma_F + \delta'_4(\sigma_A+\sigma_C)(\sigma_B+\sigma_D) + \delta'_5(\sigma_A+\sigma_C)(\sigma_E+\sigma_F) \\
&\quad + \delta'_6(\sigma_B+\sigma_D)(\sigma_E+\sigma_F) + \delta'_7\sigma_E(\sigma_A+\sigma_C)(\sigma_B+\sigma_D) + \delta'_8\sigma_A\sigma_C(\sigma_B+\sigma_D) \\
&\quad + \delta'_9\sigma_B\sigma_D(\sigma_A+\sigma_C) + \delta'_{10}\sigma_E(\sigma_A\sigma_C+\sigma_B\sigma_D).
\end{aligned} \tag{2.8}$$

In equation (2.7), the part of the contribution of $O(\varepsilon)\hat{\varepsilon}_i$ $(i = 1\text{-}13)$ arises from both the potential shifting and the coupling constants $\{\varepsilon\gamma_i\}$ of $O(\varepsilon)$. The explicit forms of $\hat{\varepsilon}_i$ are shown in table 1. Due to the symmetry in the y and z directions, some coupling constants remain equal:

$$\beta'_A = \beta'_B \qquad \delta'_1 = \delta'_2 \qquad \delta'_5 = \delta'_6 \qquad \delta'_8 = \delta'_9 \tag{2.9}$$

so nine different coupling constants appear at this stage.

In the next step, the dilatation R_y in the y direction is performed. The decimation after the potential shifting of the type $(1-\varepsilon)\beta'_B\Delta_y$ provides thirteen coupling constants.

Table 1. The coefficients a_{i0}, $a_{i\alpha}$ ($\alpha = 1$-4) of $\hat{\varepsilon}_i$; $\hat{\varepsilon}_i = a_{i0}\varepsilon\beta_0 + \Sigma_\alpha a_{i\alpha} \tanh \varepsilon\gamma_\alpha$, where β_0 is the O(1) part of β. a_{i0} and $a_{i\alpha}$ denote the contributions from the potential shifting and coupling constants of $O(\varepsilon)$ and $\{\varepsilon\gamma_\alpha\}$, respectively.

	$\varepsilon\beta_0$	$\tanh \varepsilon\gamma_1$	$\tanh \varepsilon\gamma_2$	$\tanh \varepsilon\gamma_3$	$\tanh \varepsilon\gamma_4$
$\hat{\varepsilon}_1$	0	$2u_4$	$4u_2$	0	$6u_3$
$\hat{\varepsilon}_2 = \hat{\varepsilon}_1$					
$\hat{\varepsilon}_3$	u_1	0	$4u_6$	$4u_2$	$2u_2$
$\hat{\varepsilon}_4$	0	$2u_4$	0	0	$4u_3$
$\hat{\varepsilon}_5 = \hat{\varepsilon}_4$					
$\hat{\varepsilon}_6$	0	$2u_1$	0	0	0
$\hat{\varepsilon}_7$	0	$2u_1$	$2u_2$	0	$4u_3$
$\hat{\varepsilon}_8$	u_1	0	$2u_3 + u_5$	$2u_2$	u_2
$\hat{\varepsilon}_9 = \hat{\varepsilon}_8$					
$\hat{\varepsilon}_{10}$	u_1	0	$4u_3$	$2u_2$	0
$\hat{\varepsilon}_{11}$	0	$2u_1$	0	0	$2u_3$
$\hat{\varepsilon}_{12} = \hat{\varepsilon}_{11}$					
$\hat{\varepsilon}_{13}$	u_1	0	$4u_3$	0	$2u_2$

$u_1 = p^4$, $u_2 = p^2(1+p^2)^2$, $u_3 = p^3(1+p^2)$, $u_4 = p^2(1+3p^2+p^4)$, $u_5 = p(1+p^2)(1+4p^2+p^4)$, $u_6 = p(1+p^2)(1+3p^2+p^4)$, $p = \tanh \beta_0$.

They are given by

$$\begin{aligned}
&\tanh \beta''_A = \tanh^2 \beta'_A + K_y(1 - \tanh^4 \beta'_A)\tilde{\varepsilon}_1,\\
&\tanh \beta''_B = \tanh[(2-\varepsilon)\beta'_A] + K_y(1 - \tanh^2[(2-\varepsilon)\beta'_A])\tilde{\varepsilon}_2\\
&\tanh \beta''_E = \tanh^2 \beta'_E + K_y(1 - \tanh^4 \beta'_E)\tilde{\varepsilon}_3 \qquad (2.10)\\
&\tanh \delta''_\alpha = K_y\tilde{\varepsilon}_{\alpha+3} \qquad (\alpha = 1\text{-}10)\\
&K_y = [(1+\tanh^4 \beta)^2(1+\tanh^2 2\beta)^2]^{-1}
\end{aligned}$$

where the relative positions of the plaquettes A-F on the rectangular parallelepipeds are preserved by the dilatations. The $\tilde{\varepsilon}_i$ ($i = 1$-13) can be given in the form $\tilde{\varepsilon}_i = b_{i0}\varepsilon\beta'_{B0} + \Sigma_\alpha b_{i\alpha} \tanh \delta'_\alpha$, where β'_{B0} is the O(1) part of the coupling constant β'_B. The explicit forms of b_{i0}, $\{b_{i\alpha}\}$ are shown in table 2.

In the third step, the dilatation R_z in the z direction provides the transformations

$$\begin{aligned}
&\tanh \beta'''_A = \tanh[(2-\varepsilon)\beta''_A] + K_z\{1 - \tanh^2[(2-\varepsilon)\beta''_A]\}\mathring{\varepsilon}_1\\
&\tanh \beta'''_B = \tanh^2 \beta''_B + K_z(1 - \tanh^4 \beta''_B)\mathring{\varepsilon}_2\\
&\tanh \beta'''_E = \tanh^2 \beta''_E + K_z(1 - \tanh^4 \beta''_E)\mathring{\varepsilon}_3 \qquad (2.11)\\
&\tanh \delta'''_\alpha = K_z\mathring{\varepsilon}_{\alpha+3} \qquad (\alpha = 1\text{-}10)\\
&K_z = 1/\{1 + \tanh^2[2 \tanh^{-1}(\tanh^2 \beta)]\}^2(1+\tanh^4 2\beta)^2.
\end{aligned}$$

The $\mathring{\varepsilon}_i$ are given by $\mathring{\varepsilon}_i = c_{i0}\varepsilon\beta''_{A0} + \Sigma_\alpha c_{i\alpha} \tanh \delta''_\alpha$, where β''_{A0} denotes the O(1) part of β''_A. See table 3 for c_{i0} and $c_{i\alpha}$.

Thus we arrive at an effective theory on the cube with double size. However, the approximation is not able to recover the isotropy and we still have thirteen different

Table 2. The coefficients b_{i0}, $b_{i\alpha}$ of $\tilde{\varepsilon}_i$; $\tilde{\varepsilon}_i = b_{i0}\varepsilon\beta'_{B0} + \Sigma_\alpha b_{i\alpha} \tanh \delta'_\alpha$. β'_{A0}, β'_{B0} and β'_{E0} are the O(1) parts of β'_A, β'_B and β'_E, respectively.

	$\varepsilon\beta'_{B0}$	$\tanh\delta'_1$	$\tanh\delta'_3$	$\tanh\delta'_4$	$\tanh\delta'_5$	$\tanh\delta'_7$	$\tanh\delta'_8$	$\tanh\delta'_{10}$
$\tilde{\varepsilon}_1$	0	$2(v_1+v_5)$	0	0	$4v_4$	0	$2v_3$	$4v_2$
$\tilde{\varepsilon}_2$	v_1	0	0	$2v_7$	$2v_8$	$4v_4$	v_5+v_6	0
$\tilde{\varepsilon}_3$	0	$2v_1$	$2v_6$	0	$4v_4$	0	$4v_3$	$2v_2$
$\tilde{\varepsilon}_4$	0	$2(v_1+v_5)$	0	0	0	0	0	$4v_2$
$\tilde{\varepsilon}_5$	0	$2v_1$	0	0	0	0	0	0
$\tilde{\varepsilon}_6$	0	$2v_1$	$2v_6$	0	0	0	$4v_3$	0
$\tilde{\varepsilon}_7$	v_1	0	0	v_9	$2v_2$	$2v_4$	v_5	0
$\tilde{\varepsilon}_8$	0	$2v_1$	0	0	$2v_4$	0	$2v_3$	$2v_2$
$\tilde{\varepsilon}_9$	v_1	0	0	$2v_3$	v_{10}	$2v_4$	v_6	0
$\tilde{\varepsilon}_{10}$	v_1	0	0	$2v_3$	$2v_2$	$2v_4$	0	0
$\tilde{\varepsilon}_{11}$	v_1	0	0	$2v_3$	$2v_2$	0	v_5+v_6	0
$\tilde{\varepsilon}_{12}$	0	$2v_1$	0	0	0	0	$2v_3$	0
$\tilde{\varepsilon}_{13}$	0	$2v_1$	0	0	0	0	0	$2v_2$

$v_1 = p_A^2 p_E^2$
$v_2 = p_A^2 p_E(1+p_E^2)$
$v_3 = p_A(1+p_A^2)p_E^2$
$v_4 = p_A(1+p_A^2)p_E(1+p_E^2)$
$v_5 = p_A^2(1+p_E^2)^2$
$p_A = \tanh\beta'_{A0}$

$v_6 = (1+p_A^2)^2 p_E^2$
$v_7 = p_A(1+p_A^2)(1+3p_E^2+p_E^4)$
$v_8 = (1+3p_A^2+p_A^4)p_E(1+p_E^2)$
$v_9 = p_A(1+p_A^2)(1+4p_E^2+p_E^4)$
$v_{10} = (1+4p_A^2+p_A^4)p_E(1+p_E^2)$
$p_E = \tanh\beta'_{E0}$

Table 3. The coefficients c_{i0}, $c_{i\alpha}$ of $\mathring{\varepsilon}_i$; $\mathring{\varepsilon}_i = c_{i0}\varepsilon\beta''_{A0} + \Sigma_\alpha c_{i\alpha} \tanh\delta''_\alpha$. $\{w_i\}$ are obtained from $\{v_i\}$ in table 2 by the change of variables $w_i = v_i|_{p_A=q_E, p_E=q_B}$, where $q_E = \tanh\beta''_{E0}$ and $q_B = \tanh\beta''_{B0}$. β''_{A0}, β''_{B0} and β''_{E0} are the O(1) parts of β''_A, β''_B and β''_E, respectively.

	$\varepsilon\beta''_{A0}$	$\tanh\delta''_1$	$\tanh\delta''_2$	$\tanh\delta''_3$	$\tanh\delta''_4$	$\tanh\delta''_5$	$\tanh\delta''_6$	$\tanh\delta''_7$	$\tanh\delta''_8$	$\tanh\delta''_9$	$\tanh\delta''_{10}$
$\mathring{\varepsilon}_1$	w_1	0	0	0	$2w_8$	$2w_7$	0	$4w_4$	0	w_5+w_6	0
$\mathring{\varepsilon}_2$	0	$2w_1$	$2w_6$	0	0	0	$4w_4$	0	$2w_2$	0	$4w_3$
$\mathring{\varepsilon}_3$	0	$2w_1$	0	$2w_5$	0	0	$4w_4$	0	$4w_2$	0	$2w_3$
$\mathring{\varepsilon}_4$	0	$2w_1$	0	0	0	0	0	0	0	0	0
$\mathring{\varepsilon}_5$	0	$2w_1$	$2w_6$	0	0	0	0	0	0	0	$4w_3$
$\mathring{\varepsilon}_6$	0	$2w_1$	0	$2w_5$	0	0	0	0	$4w_2$	0	0
$\mathring{\varepsilon}_7$	w_1	0	0	0	w_{10}	$2w_3$	0	$2w_4$	0	w_6	0
$\mathring{\varepsilon}_8$	w_1	0	0	0	$2w_2$	w_9	0	$2w_4$	0	w_5	0
$\mathring{\varepsilon}_9$	0	$2w_1$	0	0	0	0	$2w_4$	0	$2w_2$	0	$2w_3$
$\mathring{\varepsilon}_{10}$	w_1	0	0	0	$2w_2$	$2w_3$	0	$2w_4$	0	0	0
$\mathring{\varepsilon}_{11}$	0	$2w_1$	0	0	0	0	0	0	$2w_2$	0	0
$\mathring{\varepsilon}_{12}$	w_1	0	0	0	$2w_2$	$2w_3$	0	0	0	w_5+w_6	0
$\mathring{\varepsilon}_{13}$	0	$2w_1$	0	0	0	0	0	0	0	0	$2w_3$

coupling constants. To recover the isotropy, we take the simple average as

$$\begin{aligned}
\beta' &= \tfrac{1}{3}(\beta'''_A + \beta'''_B + \beta'''_E) \\
\varepsilon\gamma'_1 &= \tfrac{1}{3}(\delta'''_1 + \delta'''_2 + \delta'''_3) \\
\varepsilon\gamma'_2 &= \tfrac{1}{3}(\delta'''_4 + \delta'''_5 + \delta'''_6) \\
\varepsilon\gamma'_3 &= \delta'''_7 \\
\varepsilon\gamma'_4 &= \tfrac{1}{3}(\delta'''_8 + \delta'''_9 + \delta'''_{10})
\end{aligned} \tag{2.12}$$

and we arrive again at five different couplings.

Table 4. Numerical values of b_0, $\{b_i\}$, A_{00}, $\{A_{0i}\}$ and $\{A_{ij}\}$ at the MK fixed point $\beta_{MK} = 1.0044$.

b_0	-0.395
b_1	1.10×10^{-2}
b_2	8.86×10^{-2}
b_3	7.80×10^{-2}
b_4	2.12×10^{-2}
A_{00}	1.861
A_{01}	1.225
A_{02}	4.913
A_{03}	1.694
A_{04}	2.537

$$\{A_{ij}\} = \begin{pmatrix} 0.882 & 0.801 & 0.145 & 1.382 \\ 1.057 & 7.299 & 3.511 & 3.766 \\ 0.531 & 5.193 & 2.644 & 2.319 \\ 0.589 & 1.388 & 0.476 & 1.309 \end{pmatrix} \times 10^{-1}$$

If β is formally developed as $\beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 \ldots$ the recursion equations become

$$\beta_0' = F_{MK}(\beta_0) \tag{2.13}$$

at O(1) and

$$\begin{pmatrix} \beta_1' \\ \gamma_1' \\ \gamma_2' \\ \gamma_3' \\ \gamma_4' \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \beta_0 + \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & A_{04} \\ 0 & A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \tag{2.14}$$

at $O(\varepsilon)$, where b_0, $\{b_i\}$, A_{00}, $\{A_{0i}\}$ and $\{A_{ij}\}$ are complicated functions of β_0.

The recursion equation (2.13) is nothing more than the MK one, and $F_{MK}(\beta_0)$ is given by

$$F_{MK}(\beta_0) = \tfrac{1}{3}\{2\tanh^{-1} z_0^4 + \tanh^{-1}[\tanh^2(2\tanh^{-1} z_0^2)] + \tanh^{-1}(\tanh^4 2\beta_0)\} \qquad z_0 = \tanh\beta_0. \tag{2.15}$$

Equations (2.13) and (2.15) show a single non-trivial critical point at

$$\beta_C = 1.0044 (\equiv \beta_{MK}). \tag{2.16}$$

To see the correction to β_{MK} we should substitute the value of β_{MK} into the coefficients b_0, $\{b_i\}$, A_{00}, $\{A_{0j}\}$ and $\{A_{ij}\}$ in (2.14). Their numerical values are shown in table 4.

In the following section, we use (2.14) to investigate the correction in the critical properties.

3. Results and remarks

The recursion equations for $\{\gamma_i\}$ in (2.14) are independent of β_1. This means that the critical behaviour of the canonical model with a coupling constant β (2.1) can be determined simply by tracing the movement of the β_1 component of the renormalisation

group trajectories which move in the five-dimensional space. We find numerically that the correction to the critical point is

$$\beta_{1C} = -0.475 \tag{3.1}$$

and thus leads to the critical value $\beta_C = 1.004 - 0.475\varepsilon$. If we extrapolate this value to the $\varepsilon = 1$ limit following [7], we obtain

$$\beta_C = \beta_{MK} + \varepsilon(1 - \tfrac{1}{2}\varepsilon)\beta_{1C}|_{\varepsilon=1} = 0.767. \tag{3.2}$$

The value (3.2) is in remarkable agreement with the value $\beta_C = 0.761$, which is derived from that of the three-dimensional Ising model through the duality argument.

On the other hand, we find that the correction to the fixed point becomes rather large:

$$(\beta_1^*, \{\gamma_i^*\}) = (11.973, -0.176, -1.473, -1.034, -0.279). \tag{3.3}$$

As stated in [8], this is due to the fact that one of the eigenvalues of the recursion equations becomes marginal. In fact, in this case, the 4×4 submatrix $\{A_{ij}\}$ in (2.14) gives the following eigenvalues

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 1.065 \\ 0.138 \\ 0.009 \\ 0.001 \end{pmatrix} \tag{3.4}$$

from which we see that the largest one λ_1 is almost marginal. (Note that the largest eigenvalue of the full matrix is $\lambda_0 = A_{00} = 1.861$). The right and left eigenvectors for λ_1 are $(-0.989, 0.014, 0.116, 0.083, 0.022)$ and $(0, 0.134, 0.803, 0.381, 0.438)$, respectively. The phenomenon that the critical point has good convergence properties, while the fixed point does not is not new—it also occurs in the 2D Potts model [8].

An undesirable feature is that the critical point β_{1C} is located in the direction associated with the relevant eigenvalue λ_1. In other words, the fixed point (3.3) no longer governs the critical behaviour of the canonical model (2.1) but is replaced by a fixed point located at infinity. This feature makes it difficult to decide whether the critical point is of first or second order.

It is important to remark about the difference between the behaviour of the β_1^* and of the critical point. Following [11], we looked at the x_i dependence of each, where x_i are the parameters which change the strength of the potential shifting by the amount $\varepsilon x_i \gamma_i$. The value of β_1^* is very sensitive to the value of x_i, which reflects the change of the second largest eigenvalue λ_1 from the relevant to the irrelevant value passing through the marginal point. On the other hand, the critical point is almost insensitive of the change of λ_1 and shows a continuous and slow change on x_i. The details will be reported in a subsequent paper [12].

Finally, we comment that the calculation is intricate due to the appearance of the large number of coupling constants. It is possible that these complications would be reduced by using a FCC lattice instead of a cubic one, because in two dimensions they are reduced by using a triangular lattice instead of a square, but we have not investigated this possibility.

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